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Approximation in a Banach Space Defined by a Continuous Field of Banach Spaces

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1. INTRODUCTION

A very elegant, now classical, theory exists for the problem of best approximation in the space of real valued, continuous functions $C_R[a, b]$ by elements of an n-dimensional Haar subspace [3]. This theory includes the alternation theorem, the strong unicity theorem, the Freud theorem, and others. Several effective algorithms for computing best approximations in $C_R[a, b]$ are available; for example, the second algorithm of Remes.

Unfortunately, most of this elegant theory does not extend even to the best approximation problem in $C_R(D)$, where D is a rectangle in R_2.

A number of papers have considered settings that do emit best approximation results paralleling those obtainable in the space $C_R[a, b]$, see, for example [1, 10, 11, 12, 13] and the references of [11].

The focus of the present paper is the multivariate product approximation scheme introduced by Weinstein [12, 13] and subsequently considered in the linear case in [7, 8]. For nonlinear product approximation methods, see [2, 6, 7] and the references contained in [7]. The best product approximation setting has yielded several theorems in multivariate approximation that are not possible for classical multivariate best approximations (see [8, 12, 13] and, in particular, Theorem 4 in [8]). Furthermore, algorithms for computing best product approximations have proved to be very efficient when compared to know algorithms for computing classical multivariate best approximations in $C_R(D)$, [6, 7, 12].

The papers [2, 6, 7, 8, 12] have considered uniform product approximation either on rectangles or appropriate discrete sets contained in R_2. Weinstein [13] has considered a type of L^p product approximation, $1 \leqslant p \leqslant \infty$, on

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more complicated domains in R_2. The admissibility of acceptable domains is based on rather technical conditions.

In the present paper the authors view product approximation in a Banach space defined by a continuous field of Banach spaces; this abstract setting reveals basic product approximation features on more complicated domains.

Although the emphasis of the present paper is on product approximation in a Banach space defined by a continuous field of Banach spaces, other examples of the theory developed will be given.

2. Approximation in a Banach Space Defined by a Continuous Field of Banach Spaces

This section contains most of the fundamental theorems of this paper. Necessary definitions and terminology precede these theorems.

Let T be a topological space. The first definition is given in [5].

Definition 1. A continuous field $\mathscr{E}$ of Banach spaces on T is a family $(\mathbf{Z}(t))_{t\in T}$ of complex Banach spaces, together with a subset Γ of the cartesian product $\prod_{t\in T}\mathbf{Z}(t)$, such that

(a) Γ is a complex linear space of $\prod_{t\in T}\mathbf{Z}(t)$;

(b) for all $t\in T$, the set $\{\mathbf{z}(t)\colon \mathbf{z}\in\Gamma\}$ is dense in $\mathbf{Z}(t)$;

(c) for all $\mathbf{z}\in\Gamma$, the function $t\to\|\mathbf{z}(t)\|_t$ is continuous ($\|\ \|_t$ is the norm on $\mathbf{Z}(t)$);

(d) let $\mathbf{z}\in\prod_{t\in T}\mathbf{Z}(t)$; if for all $t\in T$ and all $\epsilon>0$, there is a $\mathbf{v}\in\Gamma$ such that $\|\mathbf{z}(s)-\mathbf{v}(s)\|_s\leqslant\epsilon$ for every s in some neighborhood of t, then $\mathbf{z}\in\Gamma$.

Hereafter the continuous field of Banach spaces $\mathscr{E}$ is denoted by $\mathscr{E}=((\mathbf{Z}(t))_{t\in T},\Gamma)$.

Now let $\mathbf{Z}$ be the set of all $\mathbf{z}\in\Gamma$ such that $\|\mathbf{z}(t)\|_t$ vanishes at infinity. Clearly $\mathbf{Z}$ is a linear subspace of Γ. For $\mathbf{z}\in\mathbf{Z}$, define

$$|||\,\mathbf{z}\,|||_\infty=\sup\{\|\mathbf{z}(t)\|_t\colon t\in T\}. \tag{2.1}$$

The map $\mathbf{z}\to|||\,\mathbf{z}\,|||_\infty$ defines a norm for $\mathbf{Z}$, and the pair

$$(\mathbf{Z},|||\cdot|||_\infty)\equiv\mathbf{Z}_\infty$$

is called the Banach space defined by the continuous field of Banach spaces $\mathscr{E}$. The Banach space $\mathbf{Z}_\infty$ will generally provide the setting for subsequent discussions.

Throughout the remainder of the paper T is assumed to be a compact Hausdorff space, and μ is a finite, positive, regular Borel measure defined on T with support T. We note when T is a compact Hausdorff space that $\Gamma = \mathbf{Z}$.

Let $1 \leqslant p < +\infty$, and let $\mathbf{z} \in \mathbf{Z}$. Corresponding to (2.1), define

$$|||\, \mathbf{z} \,|||_p = \left(\int_T \| \mathbf{z}(t) \|_t^p \, d\mu \right)^{1/p}. \tag{2.2}$$

Then $\mathbf{Z}_p$ denotes the normed linear space $(\mathbf{Z}, |||\cdot|||_p)$.

For additional information about continuous fields of Banach spaces the interested reader is referred to [5, pp. 186–222].

Now let $\mathbf{A}$ be a Banach algebra with norm $\|\cdot\|_{\mathbf{A}}$.

DEFINITION 2. A Banach space $\mathbf{Y}$ is said to be a Banach $\mathbf{A}$-module if $\mathbf{Y}$ is module (left or right) in the usual algebraic sense and if for all $a \in \mathbf{A}$ and $y \in \mathbf{Y}$, $\| ay \|_{\mathbf{Y}} \leqslant k \| a \|_{\mathbf{A}} \| y \|_{\mathbf{Y}}$ where k is a fixed, positive number.

By virtue of [5, 10.1.9], $\mathbf{Z}_\infty$ is a Banach $C(T)$-module, where $C(T)$ denotes the complex valued continuous functions defined on T. The reader is referred to [9] for additional properties of Banach modules.

DEFINITION 3. A sub-$\mathbf{A}$-module $\mathbf{V} \subseteq \mathbf{Y}$ is said to be free, with generator $\mathbf{G} \subseteq \mathbf{V}$, if

(a) $\text{span}_{\mathbf{A}}\, \mathbf{G} = \mathbf{V}$; and if

(b) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbf{G}$ and $a_1, \ldots, a_n$ in $\mathbf{A}$ are such that $\sum_{i=1}^n a_i \mathbf{v}_i = 0$, then $a_i = 0$, $i = 1, \ldots, n$. If in addition to (a) and (b), $\mathbf{G}$ is a finite set and $\mathbf{V}$ is closed, then $\mathbf{V}$ is said to be a finitely generated free and complete $\mathbf{A}$-module, and the elements of $\mathbf{G}$ are called free generators of $\mathbf{V}$.

DEFINITION 4. Let $\mathbf{z} \in \mathbf{Z}$. The elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{Z}$ are said to form a $C(T)$-module Chebyshev system for $\mathbf{z}$ if for each $t \in T$, $\dim \text{span}\langle \mathbf{v}_1(t), \mathbf{v}_2(t), \ldots, \mathbf{v}_n(t) \rangle = n$ and $\mathbf{z}(t)$ has a unique best approximation from $\text{span}\langle \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t) \rangle$.

We note from Definition 4 that if for each $\mathbf{z} \in \mathbf{Z}$, $\mathbf{z}(t)$ has a unique best approximation from $\text{span}\langle \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t) \rangle$, then $\text{span}\langle \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t) \rangle$ would form a Chebyshev subspace in the usual sense in $\mathbf{Z}(t)$ [11, p. 103].

The following lemma will be utilized in the proof of Theorem 1.

LEMMA. *Let $\mathbf{z} \in \mathbf{Z}_\infty$ and suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are elements of $\mathbf{Z}_\infty$ that form a $C(T)$-module Chebyshev system for $\mathbf{z}$. Let $f\colon T \to R$ be the function defined by*

$$f(t) = \inf \left(\left\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(t) \right\|_t : \alpha_i \textit{ are complex scalars}, \sum_{i=1}^n | \alpha_i | = 1 \right)$$

and set $\beta = \inf\{f(t): t \in T\}$. *Then* (a) $\beta > 0$; (b) *if* $a_1, \ldots, a_n$ *belong to* $C(T)$, *then* $|||\sum_{i=1}^n a_i \mathbf{v}_i|||_\infty \geqslant (\beta/n) \sum_{i=1}^n \| a_i \|_\infty$; (c) *the space* $\mathrm{span}_{C(T)}\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\rangle$ *is a free and complete sub-*$C(T)$*-module of* $\mathbf{Z}_\infty$ *with free generators* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Proof. Suppose $\beta = 0$. Then there is a net $\{t_{q \in \Lambda}\}$ in T such that $f(t_q) \to 0$. Consequently, for each $q \in \Lambda$ there are scalars $\alpha_{q,1}, \alpha_{q,2}, \ldots, \alpha_{q,n}$ such that $\sum_{i=1}^n | \alpha_{q,i} | = 1$ and such that

$$\lim_q \Big\| \sum_{i=1}^n \alpha_{q,i} \mathbf{v}_i(t_q) \Big\|_{t_q} = 0. \tag{2.3}$$

Since T is compact and $\{\alpha_{q,i}\}_{q \in \Lambda}$ is a bounded set we can assume, by dropping to a subset if necessary, that $t_q \to \bar{t} \in T$ and $\alpha_{q,i} \to \alpha_i$, where $\sum_{i=1}^n | \alpha_i | = 1$. It follows that

$$\begin{aligned}
\Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(\bar{t}) \Big\|_{\bar{t}} &\leqslant \Big| \Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(\bar{t}) \Big\|_{\bar{t}} - \Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(t_q) \Big\|_{t_q} \Big| \\
&\quad + \Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(t_q) \Big\|_{t_q} \\
&\leqslant \Big| \Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(\bar{t}) \Big\|_{\bar{t}} - \Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(t_q) \Big\|_{t_q} \Big| \\
&\quad + \sum_{i=1}^n | \alpha_i - \alpha_{q,i} | \, \| \mathbf{v}_i(t_q) \|_{t_q} + \Big\| \sum_{i=1}^n \alpha_{q,i} \mathbf{v}_i(t_q) \Big\|_{t_q} .
\end{aligned}$$

Now part (c) of Definition 1 and (2.3) imply that

$$\Big\| \sum_{i=1}^n \alpha_i \mathbf{v}_i(\bar{t}) \Big\|_{\bar{t}} = 0. \tag{2.4}$$

But since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a $C(T)$-module Chebyshev system for $\mathbf{z}$, $\dim \mathrm{span}\langle \mathbf{v}_1(\bar{t}), \ldots, \mathbf{v}_n(\bar{t})\rangle = n$; consequently (2.4) implies that $\alpha_i = 0$, $i = 1, \ldots, n$. But this contradicts $\sum_{i=1}^n | \alpha_i | = 1$, and hence (a) is established.

Now let $t \in T$ and let $a_1, a_2, \ldots, a_n$ belong to $C(T)$. Clearly part (a) of the Lemma implies that

$$\beta \mid a_i(t) | \leqslant \Big\| \sum_{i=1}^n a_i(t) \, \mathbf{v}_i(t) \Big\|_t \leqslant \Big|\Big|\Big| \sum_{i=1}^n a_i \mathbf{v}_i \Big|\Big|\Big|_\infty . \tag{2.5}$$

Thus $\beta \| a_i \|_\infty \leqslant ||| \sum_{i=1}^n a_i \mathbf{v}_i |||_\infty$, and this inequality implies (b).

To establish part (c) we need to show that $\text{span}_{C(T)}\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle$ is complete. Suppose $\{\mathbf{g}_j\}_{j=1}^{\infty} \subseteq \text{span}_{C(T)}\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle$ is a Cauchy sequence. Now

$$
\begin{aligned}
|||\, \mathbf{g}_j - \mathbf{g}_k \,|||_\infty &= \left|\left|\left| \sum_{i=1}^{n} (a_{ij} - a_{ik})\, \mathbf{v}_i \right|\right|\right|_\infty \\
&\geqslant (\beta/n) \sum_{i=1}^{n} \| a_{ij} - a_{ik} \|_\infty
\end{aligned}
$$

by part (b) of the Lemma. Thus $\{a_{ij}\}_{j=1}^{\infty}$ is Cauchy in $C(T)$. Part (c) now follows from the completeness of $C(T)$ and $\mathbf{Z}_\infty$. ∎

THEOREM 1. *Let $\mathbf{z} \in \mathbf{Z}_\infty$ and suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are elements in $\mathbf{Z}_\infty$ that form a $C(T)$-module Chebyshev system for $\mathbf{z}$. For each $t \in T$ let $a_1(t), \ldots, a_n(t)$ be the unique complex numbers satisfying*

$$
\begin{aligned}
&\left\| \mathbf{z}(t) - \sum_{i=1}^{n} a_i(t)\, \mathbf{v}_i(t) \right\|_t \\
&\qquad = \inf\{\| \mathbf{z}(t) - \mathbf{q}(t)\|_t \colon \mathbf{q}(t) \in \text{span}\langle \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\rangle\}. \qquad (2.6)
\end{aligned}
$$

Then each a_i, $i = 1, \ldots, n$ is an element of $C(T)$.

Proof. Let $f(t)$ and β be defined as in the Lemma. Then (2.5) implies that

$$
\begin{aligned}
\sum_{i=1}^{n} | a_i(t)| &\leqslant (n/\beta) \left\| \sum_{i=1}^{n} a_i(t)\, \mathbf{v}_i(t) \right\|_t \\
&\leqslant (n/\beta) \left\| \sum_{i=1}^{n} a_i(t)\, \mathbf{v}_i(t) - z(t) \right\|_t + (n/\beta)\, \| \mathbf{z}(t)\|_t \\
&\leqslant (2n/\beta)\, \| \mathbf{z}(t)\|_t .
\end{aligned}
$$

Thus

$$
\| a_i \|_\infty \leqslant (2n/\beta)\, |||\, \mathbf{z}\, |||_\infty , \qquad i = 1, 2, \ldots, n. \qquad (2.7)
$$

As usual ℓ_1^n denotes the Banach space of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of complex numbers with norm $\| \alpha \| = \sum_{i=1}^{n} | \alpha_i |$. Let $G\colon T \times \ell_1^n \to R$ be the mapping defined by

$$
G(t, \alpha) = \left\| \mathbf{z}(t) - \sum_{i=1}^{n} \alpha_i \mathbf{v}_i(t) \right\|_t \qquad (2.8)
$$

for each $t \in T$ and $\alpha \in \ell_1^n$. Let (s, τ) be a fixed point in $T \times \ell_1^n$. Then

$$\begin{aligned} | G(t, \alpha) - G(s, \tau)| &\leqslant | G(t, \alpha) - G(t, \tau)| + | G(t, \tau) - G(s, \tau)| \\ &\leqslant \sum_{i=1}^{n} | \alpha_i - \tau_i | \, \| \mathbf{v}_i(t) \|_t \\ &\quad + \left| \left\| \mathbf{z}(t) - \sum_{i=1}^{n} \tau_i \mathbf{v}_i(t) \right\|_t - \left\| \mathbf{z}(s) - \sum_{i=1}^{n} \tau_i \mathbf{v}_i(s) \right\|_s \right|. \end{aligned}$$

This inequality and part (c) of Definition 1 now imply that G is a continuous map. Next define $\rho: T \to R$ by the formula

$$\rho(t) = \left\| \mathbf{z}(t) - \sum_{i=1}^{n} a_i(t) \mathbf{v}_i(t) \right\|_t = G(t, a(t)),$$

where $a: T \to \ell_1^n$ is given by $a(t) = (a_1(t), a_2(t),..., a_n(t))$. Observe that

$$\rho(t) - \rho(s) \leqslant G(t, a(s)) - G(s, a(s)) \tag{2.9}$$

and that

$$\rho(s) - \rho(t) \leqslant G(s, a(t)) - G(t, a(t)). \tag{2.10}$$

The continuity of G and (2.7) imply that $| G(s, a(t)) - G(t, a(t))| \to 0$ as $t \to s$. Consequently (2.9) and (2.10) imply that ρ is continuous. Now let $\epsilon > 0$ and $s \in T$ be given. The argument given in [12, Theorem 2.2] implies that there is a $\delta > 0$ such that if $\tau = (\tau_1, ..., \tau_n) \in \ell_1^n$ has the property that $G(s, \tau) \leqslant \rho(s) + \delta$, then $\sum_{i=1}^{n} | \tau_i - a_i(s)| < \epsilon$. Choose a neighborhood U of s such that for each $t \in U$, $| \rho(t) - \rho(s)| < \delta/2$ and $| G(s, a(t)) - G(t, a(t))| < \delta/2$. It follows for $t \in U$ that

$$\begin{aligned} G(s, a(t)) &\leqslant | G(s, a(t)) - G(t, a(t))| + | \rho(t) - \rho(s)| + \rho(s) \\ &\leqslant \rho(s) + \delta. \end{aligned}$$

Thus $\sum_{i=1}^{n} | a_i(t) - a_i(s)| < \epsilon$, and consequently $a_i \in C(T)$, $i = 1, 2,..., n$. ∎

The next theorem establishes a fundamental link between the normed linear spaces $\mathbf{Z}_p$, $1 \leqslant p \leqslant \infty$, and approximation from the span of a $C(T)$-module Chebyshev system.

THEOREM 2. *Let* $\mathbf{z}$ *be a fixed element of* $\mathbf{Z}_\infty$, *and suppose that* $\mathbf{v}_1, ..., \mathbf{v}_n$ *are elements in* $\mathbf{Z}_\infty$ *that form a* $C(T)$*-module Chebyshev system for* $\mathbf{z}$. *Let* $\mathbf{V} = \operatorname{span}_{C(T)}\langle \mathbf{v}_1, ..., \mathbf{v}_n \rangle$, *and let*

$$\mathscr{L}_{\mathbf{V}}^p(\mathbf{z}) = \{\mathbf{q} \in \mathbf{V}: ||| \mathbf{z} - \mathbf{q} |||_p = \inf_{\mathbf{v} \in \mathbf{V}} ||| \mathbf{z} - \mathbf{v} |||_p\}, \qquad 1 \leqslant p \leqslant \infty.$$

Then $\mathscr{L}_{\mathbf{V}}^1(\mathbf{z})$ *contains exactly one element* $\mathbf{v}_0$; *moreover*, $\mathbf{v}_0 \in \mathscr{L}_{\mathbf{V}}^p(\mathbf{z})$ *for* $1 \leqslant p \leqslant \infty$.

Proof. Part (c) of the Lemma implies that $\mathbf{V}$ is a free and complete sub-$C(T)$-module of $\mathbf{Z}_\infty$ with free generators $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Now let $a_1, \ldots, a_n$ be the elements of $C(T)$ defined in (2.6). Let $\mathbf{v}_0 = \sum_{i=1}^n a_i \mathbf{v}_i$. Then clearly $\mathbf{v}_0 \in \mathscr{L}_{\mathbf{V}}^p(\mathbf{z})$, $1 \leqslant p \leqslant \infty$. Next suppose that $\mathbf{q} = \sum_{i=1}^n b_i \mathbf{v}_i$ is in $\mathscr{L}_{\mathbf{V}}^1(\mathbf{z})$, and assume for some $\bar{t} \in T$ that $\mathbf{q}(\bar{t}) \neq \mathbf{v}(\bar{t})$. Since $\| \mathbf{z}(\bar{t}) - \mathbf{v}_0(\bar{t})\|_{\bar{t}} < \| \mathbf{z}(\bar{t}) - \mathbf{q}(\bar{t})\|_{\bar{t}}$, there is a neighborhood U of $\bar{t}$ and a $\delta > 0$ such that

$$\| \mathbf{z}(t) - \mathbf{v}_0(t)\|_t + \delta < \| \mathbf{z}(t) - \mathbf{q}(t)\|_t \tag{2.11}$$

for all $t \in U$. It follows that

$$\begin{aligned} ||| \mathbf{z} - \mathbf{v}_0 |||_1 &= \int_T \| \mathbf{z}(t) - \mathbf{v}_0(t)\|_t \, d\mu(t) \\ &= \int_U \| \mathbf{z}(t) - \mathbf{v}_0(t)\|_t \, d\mu(t) + \int_{T \setminus U} \| \mathbf{z}(t) - \mathbf{v}_0(t)\|_t \, d\mu(t). \end{aligned}$$

This equality and (2.11) imply that

$$\begin{aligned} ||| \mathbf{z} - \mathbf{v}_0 |||_1 &\leqslant -\delta\mu(U) + \int_U \| \mathbf{z}(t) - \mathbf{q}(t)\|_t \, d\mu(t) \\ &\quad + \int_{T \setminus U} \| \mathbf{z}(t) - \mathbf{q}(t)\|_t \, d\mu(t) \\ &\leqslant -\delta\mu(U) + ||| \mathbf{z} - \mathbf{q} |||_1 \\ &= -\delta\mu(U) + ||| \mathbf{z} - \mathbf{v}_0 |||_1 . \end{aligned}$$

Therefore $\mu(U) = 0$, which contradicts the fact that the support of μ is T. ∎

Before proceeding to the next theorem we give two examples of $C(T)$-module Chebyshev systems in Banach spaces defined by continuous fields of Banach spaces. The first of these examples will be utilized in subsequent product approximation considerations.

EXAMPLE 1. Suppose that J is a compact subset of the real numbers and that $\operatorname{span}\langle f_1, f_2, \ldots, f_m\rangle \subseteq C(J)$ is a Chebyshev subspace of dimension m. Let $I = \{1, 2, \ldots, n\}$ and set $\mathbf{A}_i = C(J)$ for each $i \in I$. Define $\mathbf{f}_j \in \prod_{i=1}^n \mathbf{A}_i$ by $\mathbf{f}_j(i) = f_j$, $j = 1, 2, \ldots, m$. We now show that $\mathbf{f}_1, \ldots, \mathbf{f}_m$ form a $C(I)$-module Chebyshev system for each $\mathbf{h}$ contained in the Banach space $\mathbf{A}_\infty$ defined by the continuous field of Banach spaces $((\mathbf{A}_i)_{i \in I}, \prod_{i=1}^n \mathbf{A}_i)$. In terms of previous notation we observe that $\Gamma = \mathbf{Z} = \prod_{i=1}^{n} \mathbf{A}_i$, $T = I$, and that $\mathbf{A}_\infty = \mathbf{Z}_\infty$. Also

$$\| \mathbf{h}(i) \|_i = \sup_{y \in J} | \mathbf{h}(i)(y) | = \| \mathbf{h}(i) \|_J$$

and

$$\||\, \mathbf{h} \,\||_\infty = \max_{1 \leqslant i \leqslant n} \| \mathbf{h}(i) \|_J .$$

Clearly dim span$\langle \mathbf{f}_1(i), \ldots, \mathbf{f}_m(i) \rangle$ = dim span$\langle f_1, \ldots, f_m \rangle = m$, $i = 1, \ldots, n$. Let $\mathbf{h} \in \mathbf{A}_\infty$. Since span$\langle f_1, \ldots, f_m \rangle$ is a Chebyshev subspace in $C(J)$, each $\mathbf{h}(i)$ has a unique best approximation from this span. Thus Definition 4 implies that $\mathbf{f}_1, \ldots, \mathbf{f}_m$ is a $C(I)$-module Chebyshev system for $\mathbf{h}$. Let $c_1(i), \ldots, c_m(i)$ be the coefficients determined in (2.6). Then

$$\Big\| \mathbf{h}(i) - \sum_{j=1}^{m} c_j(i)\, \mathbf{f}_j(i) \Big\|_i = \sup_{y \in J} \Big| \mathbf{h}(i)(y) - \sum_{j=1}^{m} c_j(i)\, f_j(y) \Big|.$$

Therefore $\mathbf{f}(i)(y) = \sum_{j=1}^{m} c_j(i) f_j(y)$ is the classical unique best approximation to $\mathbf{h}(i)(y)$ on the set J from span$\langle f_1, \ldots, f_m \rangle$, $i = 1, 2, \ldots, n$.

EXAMPLE 2. Suppose $((\mathbf{Z}(t))_{t \in T}, \Gamma)$ is a continuous field of Hilbert spaces; that is, each $\mathbf{Z}(t)$ is a Hilbert space. Assume $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are elements in $\mathbf{Z}_\infty$ such that for each $t \in T$, $\{\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\}$ is an orthonormal set in $\mathbf{Z}(t)$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a $C(T)$-module Chebyshev system for any $\mathbf{f} \in \mathbf{Z}_\infty$. The reader is referred to [4] for non-trivial examples of continuous fields of Hilbert spaces.

In the next theorem the basic results needed to extend product approximation to more general domains are established.

THEOREM 3. *Let S be a compact Hausdorff space, Q a continuous mapping of S onto T, and $\mathbf{Y}$ a Banach space. Let θ be the subset of $\prod_{t \in T} C(S_t, \mathbf{Y})$, $S_t = Q^{-1}(t)$, given by $\theta = \{\hat{\mathbf{f}} : \mathbf{f} \in C(S, \mathbf{Y})\}$, where $\hat{\mathbf{f}}$ is defined by $\hat{\mathbf{f}}(t) = \mathbf{f} \mid S_t$. Then*

$$((C(S_t, \mathbf{Y}))_{t \in T}, \theta) \tag{2.12}$$

is a continuous field of Banach spaces if and only if Q is an open map. Moreover, when Q is an open map, the Banach space $\mathbf{A}_\infty$ defined by (2.12) *is isometrically isomorphic to $C(S, \mathbf{Y})$.*

Proof. First assume that Q is an open map. Then the properties of Definition 1 need to be verified.

It is clear that θ is a complex linear subspace of $\prod_{t \in T} C(S_t, \mathbf{Y})$, and, by virtue of [5, 10.1.12, p. 190], $C(S_t, \mathbf{Y}) = \{\hat{\mathbf{f}}(t) : \hat{\mathbf{f}} \in \theta\}$ for each $t \in T$. Thus properties (a) and (b) of Definition 1 are verified.

Now let $\hat{\mathbf{f}} \in \theta$, where $\mathbf{f} \in C(S, \mathbf{Y})$, and suppose $\{t_\alpha\}_{\alpha \in \Lambda}$ is a net in T that converges to t. Assume

$$\overline{\lim_\alpha} \| \hat{\mathbf{f}}(t_\alpha)\|_{t_\alpha} > \| \hat{\mathbf{f}}(t)\|_t . \tag{2.13}$$

(Here $\| \hat{\mathbf{f}}(t)\|_t = \sup_{s \in S_t} \| \mathbf{f}(s)\|_{\mathbf{Y}}$.) Suppose that $\{t_n\}$ is a subnet satisfying

$$\lim_n \| \hat{\mathbf{f}}(t_n)\|_{t_n} = \overline{\lim_\alpha} \| \hat{\mathbf{f}}(t_\alpha)\|_{t_\alpha} . \tag{2.14}$$

We may assume there is a net $\{s_n\} \subseteq S$, $s_n \in S_{t_n}$, that converges to some $s \in S$ and that satisfies

$$\| \hat{\mathbf{f}}(t_n)\|_{t_n} = \sup_{s \in S_{t_n}} \| \mathbf{f}(s)\|_{\mathbf{Y}} = \| \mathbf{f}(s_n)\|_{\mathbf{Y}} .$$

Clearly $\lim_n \| \mathbf{f}(s_n)\|_{\mathbf{Y}} = \lim_n \| \hat{\mathbf{f}}(t_n)\|_{t_n}$. Since Q is continuous and $Q^{-1}(t_n) = S_{t_n}$, $t_n \to t$ implies that $Q(s) = t$. Thus (2.13) and (2.14) imply that

$$\begin{aligned} \| \hat{\mathbf{f}}(t)\|_t &< \lim_n \| \hat{\mathbf{f}}(t_n)\|_{t_n} = \lim_n \| \mathbf{f}(s_n)\|_{\mathbf{Y}} \\ &= \| \mathbf{f}(s)\|_{\mathbf{Y}} = \| \hat{\mathbf{f}}(t)(s)\|_{\mathbf{Y}} \\ &\leqslant \sup_{s \in S_t} \| \hat{\mathbf{f}}(t)(s)\|_{\mathbf{Y}} = \| \hat{\mathbf{f}}(t)\|_t , \end{aligned}$$

which is a contradiction. Therefore

$$\overline{\lim_\alpha} \| \hat{\mathbf{f}}(t_\alpha)\|_{t_\alpha} \leqslant \| \hat{\mathbf{f}}(t)\|_t . \tag{2.15}$$

Now select $\bar{s} \in Q^{-1}(t)$ so that

$$\| \hat{\mathbf{f}}(t)\|_t = \sup_{s \in S_t} \| \hat{\mathbf{f}}(t)(s)\|_{\mathbf{Y}} = \| \hat{\mathbf{f}}(t)(\bar{s})\|_{\mathbf{Y}} = \| \mathbf{f}(\bar{s})\|_{\mathbf{Y}} .$$

Let $\epsilon > 0$ be given. By continuity of $\mathbf{f}$, the set

$$U = \{s \in S : \| \mathbf{f}(\bar{s})\|_{\mathbf{Y}} - \epsilon < \| \mathbf{f}(s)\|_{\mathbf{Y}} < \| \mathbf{f}(\bar{s})\|_{\mathbf{Y}} - \epsilon\}$$

is an open neighborhood of $\bar{s}$. Since Q is an open map and $Q(\bar{s}) = t$, it follows that $Q(U)$ is an open neighborhood of t. Thus there is an n_0 from the directed set Λ such that $t_n \in Q(U)$ for $n \geqslant n_0$. Choose s_n in $U \cap Q^{-1}(t_n)$. Note for $n \geqslant n_0$ that

$$\begin{aligned} \| \hat{\mathbf{f}}(t_n)\|_{t_n} &= \sup_{s \in S_{t_n}} \| \hat{\mathbf{f}}(t_n)(s)\|_{\mathbf{Y}} \geqslant \| \hat{\mathbf{f}}(t_n)(s_n)\|_{\mathbf{Y}} \\ &= \| \mathbf{f}(s_n)\|_{\mathbf{Y}} > \| \mathbf{f}(\bar{s})\|_{\mathbf{Y}} - \epsilon. \end{aligned}$$

Thus

$$\varliminf_{n} \| \hat{\mathbf{f}}(t_n)\|_{t_n} \geqslant \| \mathbf{f}(\bar{s})\|_{\mathbf{Y}} = \| \hat{\mathbf{f}}(t)(\bar{s})\|_{\mathbf{Y}} = \| \hat{\mathbf{f}}(t)\|_t .$$

This inequality and (2.15) now imply that

$$\lim_{\alpha} \| \hat{\mathbf{f}}(t_\alpha)\|_{t_\alpha} = \| \hat{\mathbf{f}}(t)\|_t .$$

Therefore $t \to \| \hat{\mathbf{f}}(t)\|_t$ is a continuous mapping, concluding the verification of part (c) of Definition 1.

Now let $\hat{\mathbf{g}} \in \prod_{t \in T} C(S_t, \mathbf{Y})$. Suppose for each $\epsilon > 0$ and $\bar{t} \in T$ there is an open neighborhood $V_{\bar{t}}$ of $\bar{t}$ and an $\mathbf{f}$ in $C(T, Y)$ satisfying

$$\| \hat{\mathbf{g}}(t) - \hat{\mathbf{f}}(t)\|_t < \epsilon/3, \qquad t \in V_{\bar{t}} . \tag{2.16}$$

Since $\{V_t\}_{t \in T}$ is an open cover of T, there is a finite subcover $V_{t_1}, V_{t_2}, ..., V_{t_n}$ and elements $\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, ..., \hat{\mathbf{f}}_n$ that satisfy (2.16). Let $\{e_i\}_{i=1}^n$ be a partition of unity such that e_i vanishes outside of V_{t_i}. For each $s \in S$, let $\mathbf{f}(s) = \sum_{i=1}^n e_i(Q(s))\, \mathbf{f}_i(s)$. Clearly $\mathbf{f} \in C(S, \mathbf{Y})$ and $\hat{\mathbf{f}} = \sum_{i=1}^n e_i \hat{\mathbf{f}}_i$. Note that if $t \in T$, then by (2.16)

$$\| \hat{\mathbf{g}}(t) - \hat{\mathbf{f}}(t)\|_t = \Big\| \sum_{i=1}^n e_i(t)(\hat{\mathbf{g}}(t) - \hat{\mathbf{f}}_i(t)) \Big\|_t < \epsilon/3.$$

Define $\mathbf{g}(s) = \hat{\mathbf{g}}(Q(s))(s)$ and let $\bar{s}$ be a fixed element in S. Choose an open neighborhood U of $\bar{s}$ so that $\| \mathbf{f}(s) - \mathbf{f}(\bar{s})\|_{\mathbf{Y}} < \epsilon/3$, $s \in U$. Then for $s \in U$,

$$\begin{aligned} \| \mathbf{g}(s) - \mathbf{g}(\bar{s})\|_{\mathbf{Y}} &= \| \hat{\mathbf{g}}(Q(s))(s) - \hat{\mathbf{g}}(Q(\bar{s}))(\bar{s})\|_{\mathbf{Y}} \\ &\leqslant \| \hat{\mathbf{g}}(Q(s)) - \hat{\mathbf{f}}(Q(s))\|_{Q(s)} \\ &\quad + \| \mathbf{f}(s) - \mathbf{f}(\bar{s})\|_{\mathbf{Y}} \\ &\quad + \| \hat{\mathbf{f}}(Q(s)) - \hat{\mathbf{g}}(Q(\bar{s}))\|_{Q(\bar{s})} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Therefore $\mathbf{g} \in C(S, \mathbf{Y})$ and consequently $\hat{\mathbf{g}} \in \theta$. Thus part (d) of Definition 1 is verified, and (2.12) is a continuous field of Banach spaces.

Since it is clear that $\mathbf{f} \to \hat{\mathbf{f}}$ is an isometric isomorphism of $C(S, \mathbf{Y})$ onto $\mathbf{A}_\infty$ when Q is an open map, to complete the proof we need to show that if (2.12) is a continuous field of Banach spaces, then Q is an open map.

Let U be an open subset of S and suppose t is a limit point of $T \backslash Q(U)$. It follows that there is a net $\{t_n\}$ in $T \backslash Q(U)$ that converges to t. Note that $Q^{-1}(t_n) = S_{t_n}$ does not meet U. Suppose $t \in Q(U)$. Choose s in U so that

$Q(s) = t$. Next choose a continuous function $\mathbf{f} \in C(S, \mathbf{Y})$ so that $\mathbf{f}(s) \neq 0$ and $\mathbf{f}$ vanishes outside of U. Then part (c) of Definition 1 implies

$$0 \neq \| \hat{\mathbf{f}}(t)\|_t = \lim_n \| \hat{\mathbf{f}}(t_n)\|_{t_n} = \lim_n \sup_{s \in S_{t_n}} \| \mathbf{f}(s)\|_{\mathbf{Y}} = 0,$$

a contradiction. Thus $t \in T \backslash Q(U)$ and consequently Q is an open map. ∎

Example 3. Let S be a compact subset of R_2, P a projection of R_2 onto the x-axis, Q a projection of R_2 onto the y-axis, and $\mathbf{H}$ a separable Hilbert space. Assume S is such that $Q: S \to R$ is an open map. Set $T = Q(S)$, and let $((C(S_t, \mathbf{H}))_{t \in T}, \theta)$ be the continuous field of Banach spaces defined as in Theorem 3. Denote by $\mathbf{A}_\infty$ the Banach space defined by this continuous field of Banach spaces. Let $\mathbf{f}$ be a fixed element of $\mathbf{A}_\infty$ and suppose that g_1, $g_2, \ldots, g_n$ are complex valued functions of a real variable defined on $P(S)$ that vanish nowhere on $P(S)$. Let $e_1, e_2, \ldots, e_n$ be orthonormal elements in $\mathbf{H}$. Now define $\mathbf{v}_i \in \mathbf{A}_\infty$ by the formula $\mathbf{v}_i(t)(s) = g_i(P(s))\, e_i$ for all $s \in S_t$, $i = 1, 2, \ldots, n$. Then by virtue of [14, p. 387], $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a $C(T)$-module Chebyshev system for every $\mathbf{f} \in \mathbf{A}_\infty$.

Remark. Recall that $\mathbf{A}_\infty$ is isometrically isomorphic to $C(S, \mathbf{H})$ and note that in general $\mathbf{A}_\infty$ is non-trivial (see [5, 10.1.4, p. 188] for the definition of a trivial continuous field of Banach spaces). The space $\mathbf{A}_\infty$ may at first appear to be a complicated copy of $C(S, \mathbf{H})$; however, for problems dealing with approximation the base space T is a critical factor (e.g., the abundance of useful theory of approximation in R versus the scarcity of useful theory in R_2).

Example 4. Let S be a compact subset of R_2 that satisfies Property 2.4 of [13]. Let P and Q be the projections of R_2 onto the x and y axis, respectively, let $\mathbf{Y}$ be a complex Banach space, and let $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$ be elements in $C(P(S), \mathbf{Y})$. Set $T = Q(S)$. Assume that for each $t \in T$, span$\langle \mathbf{g}_1 \mid P(Q^{-1}(t)), \ldots, \mathbf{g}_n \mid P(Q^{-1}(t))\rangle$ is an n-dimensional Chebyshev subspace of $C(P(Q^{-1}(t)), \mathbf{Y})$. Now Lemma 2.10 of [13], extended to complex valued functions, implies (as in the proof of Theorem 3) that Q is an open map. Thus Theorem 3 implies that

$$(C(Q^{-1}(t), \mathbf{Y}))_{t \in T}, \theta) \tag{2.17}$$

is a continuous field of Banach spaces. Let $\mathbf{A}_\infty$ denote the Banach space defined by (2.17). Define $\hat{\mathbf{v}}_i$ by

$$\hat{\mathbf{v}}_i(t) = (\mathbf{g}_i \circ P) \mid Q^{-1}(t), \qquad i = 1, 2, \ldots, n,$$

for each $t \in T$. Then $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_n$ is a $C(T)$-module Chebyshev system for all $\hat{\mathbf{f}} \in \mathbf{A}_\infty$. Given $\hat{\mathbf{f}} \in \mathbf{A}_\infty$ the approximation problem (2.6) becomes

$$\left\| \hat{\mathbf{f}}(t) - \sum_{i=1}^{n} a_i(t)\, \hat{\mathbf{v}}_i(t) \right\|_t = \sup_{s \in Q^{-1}(t)} \left\| \hat{\mathbf{f}}(t)(s) - \sum_{i=1}^{n} a_i(t)\, \hat{\mathbf{v}}_i(t)(s) \right\|_{\mathbf{Y}}$$

$$= \sup_{x \in P(Q^{-1}(t))} \left\| \mathbf{f}(x, t) - \sum_{i=1}^{n} a_i(t)\, \mathbf{g}_i(x) \right\|_{\mathbf{Y}}, \quad (2.18)$$

where $\mathbf{f} \in C(S, \mathbf{Y})$. Recall that Theorem 1 implies $a_i \in C(T)$, $i = 1, \ldots, n$.

EXAMPLE 5. Let S be a compact subset of R_2 that satisfies Property 2.5 of [13]. Let P and Q be the projections of R_2 onto the x and y axis, respectively, and set $T = Q(S)$. Assume $g_1, g_2, \ldots, g_n$ are elements in $C(P(S))$ such that span$\langle g_1 \mid P(Q^{-1}(t)), \ldots, g_n \mid P(Q^{-1}(t))\rangle$ is an n-dimensional Chebyshev subspace of $C(P(Q^{-1}(t)))$ for each $t \in T$. For each $t \in T$, let λ_t denote the counting measure on $Q^{-1}(t)$ if this set is finite or let λ_t denote Lebesgue measure if $Q^{-1}(t)$ is infinite. Assume $\lambda_t(Q^{-1}(t)) > 0$ for all $t \in T$. Let $1 < p < \infty$ and let Λ be the subset of $\prod_{t \in T} L^p(Q^{-1}(t), \lambda_t)$ defined by $\Lambda = \{\mathbf{f} : f \in C(S)\}$, where $\mathbf{f}(t) = f \mid Q^{-1}(t)$. Clearly Λ satisfies properties (a) and (b) of Definition 1 (Γ replaced by Λ). Lemma 2.10 of [13], extended to complex valued functions, implies that Λ satisfies property (c) of Definition 1. Now let Γ be the unique subset of $\prod_{t \in T} L^p(Q^{-1}(t), \lambda_t)$ that contains Λ given by [5, 10.2.3, p. 192]. Then

$$((L^p(Q^{-1}(t), \lambda_t))_{t \in T}, \Gamma) \quad (2.19)$$

is a continuous field of Banach spaces. Let $\mathbf{B}_\infty$ be the Banach space defined by (2.19). Next define $\mathbf{v}_i(t)(s) = g_i(P(s))$ for each $t \in T$ and $s \in Q^{-1}(t)$, $i = 1, 2, \ldots, n$. If $1 < p < \infty$, then for any $\mathbf{f} \in \mathbf{B}_\infty$, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a $C(T)$-module Chebyshev system for $\mathbf{f}$.

For $p = 1$ we assume for each $t \in T$ that $P(Q^{-1}(t))$ is an interval, and that $\mathbf{f} \in \mathbf{B}_\infty$ is such that $\mathbf{f}(t) = \mathbf{f} \mid Q^{-1}(t)$ is real valued for each $t \in T$. If $g_1, g_2, \ldots, g_n$ above are real valued functions, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ forms a $C(T)$-module Chebyshev system for $\mathbf{f}$. We note the possibility of non-unique best approximations in the discrete L^1 setting necessitates the requirement that $P(Q^{-1}(t))$ be an interval.

Remark. Examples 4 and 5 provide the identifications necessary to view product approximation (as examined by Weinstein in R_2 [13]) in a Banach space generated by a continuous field of Banach spaces.

We conclude the present section with a density theorem.

THEOREM 4. *Let* $\mathbf{z} \in \mathbf{Z}_\infty$ *and suppose* $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ *is a sequence in* $\mathbf{Z}_\infty$ *such that for each positive integer* n, *the elements* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *form a* $C(T)$-*module Chebyshev system for* $\mathbf{z}$. *Moreover, assume for each* $t \in T$ *that* $\text{span}\langle \mathbf{v}_1(t), \mathbf{v}_2(t), \ldots \rangle$ *is dense in* $\mathbf{Z}(t)$. *Also assume for each positive integer n that* $\mathbf{q}_n$ *is the unique element in* $\mathscr{L}^1_{\mathbf{V}_n}(\mathbf{z})$ *described in Theorem* 2, *where* $\mathbf{V}_n = \text{span}_{C(T)}\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. *Then* $\lim_{n\to\infty} |||\, \mathbf{z} - \mathbf{q}_n \,|||_\infty = 0$.

Proof. Let $\epsilon > 0$ be given. For each $t \in T$ there is a positive integer n_t such that $\| \mathbf{z}(t) - \mathbf{q}_{n_t}(t) \|_t < \epsilon$. Now let

$$U_t = \{ s \in T : \| \mathbf{z}(s) - \mathbf{q}_{n_t}(s) \|_s < \epsilon \}.$$

Clearly $\{U_t\}_{t\in T}$ is an open cover of T. Therefore compactness of T implies there is a finite subcover $U_{t_1}, U_{t_2}, \ldots, U_{t_k}$ of T. If $N = \max\{n_{t_1}, n_{t_2}, \ldots, n_{t_k}\}$, then $|||\, \mathbf{z} - \mathbf{q}_N \,|||_\infty < \epsilon$. ∎

3. Product Approximation in a Banach Space Defined by a Continuous Field of Banach Spaces

Let T be a compact Hausdorff space, $((\mathbf{Z}(t))_{t\in T}, \Gamma)$ a continuous field of Banach spaces, and $\mathbf{Z}_p$, $1 \leqslant p \leqslant \infty$ the associated normed linear space defined below (2.2). The next result is a variant of a theorem due to Weinstein [12, Theorem 2.2] and was first observed in the setting of Example 4 (for real functions and S a rectangle) by Henry and Schmidt [8, Lemma 1].

THEOREM 5. *Let* $\mathbf{z}$ *be a fixed element in* $\mathbf{Z}_\infty$, *and suppose* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *are elements in* $\mathbf{Z}_\infty$ *that form a* $C(T)$-*module Chebyshev system for* $\mathbf{z}$. *Let* $a_1, a_2, \ldots, a_n$ *be the unique elements in* $C(T)$ *that generate the singleton* $\sum_{i=1}^n a_i \mathbf{v}_i$ *in* $\mathscr{L}_{\mathbf{V}}^1(\mathbf{z})$, *where* $\mathbf{V} = \text{span}_{C(T)}\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. *Set*

$$\rho(t) = \left\| \mathbf{z}(t) - \sum_{i=1}^n a_i(t)\, \mathbf{v}_i(t) \right\|_t,$$

and suppose $b_i \in C(T)$, $i = 1, 2, \ldots, n$. *Then for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\left\| \mathbf{z}(t) - \sum_{i=1}^n b_i(t)\, \mathbf{v}_i(t) \right\|_t < \rho(t) + \delta$$

for all $t \in T$ *implies that*

$$\sum_{i=1}^n | a_i(t) - b_i(t) | < \epsilon$$

for all $t \in T$.

Proof. Suppose the conclusion is false. Then there exists an $\epsilon > 0$ such that for each positive integer m there is an element $\sum_{i=1}^{n} b_{m,i}\mathbf{v}_i$ in $\mathbf{V}$ and an element $t_m \in T$ such that

$$\left\| \mathbf{z}(t) - \sum_{i=1}^{n} b_{m,i}(t)\, \mathbf{v}_i(t) \right\|_t < \rho(t) + \frac{1}{m} \tag{3.1}$$

for all $t \in T$ and such that $\sum_{i=1}^{n} | a_i(t_m) - b_{m,i}(t_m)| \geqslant \epsilon$. Since T is compact, we may assume without loss of generality that $\{t_m\}_{m=1}^{\infty}$ converges to some $\bar{t} \in T$. The Lemma and (3.1) imply that $\{\| b_{m,i} \|_\infty\}_{m=1}^{\infty}$ is a bounded set. Thus we may assume without loss of generality that $b_{m,i}(t_m) \to \alpha_i$, α_i a complex scalar, $i = 1, 2,..., n$. Therefore, since $G(t, \alpha)$ and $\rho(t)$ are continuous (see the proof of Theorem 1) we have from (3.1) that

$$\left\| \mathbf{z}(\bar{t}) - \sum_{i=1}^{n} \alpha_i \mathbf{v}_i(\bar{t}) \right\|_{\bar{t}} \leqslant \rho(\bar{t}). \tag{3.2}$$

Moreover,

$$\sum_{i=1}^{n} | a_i(\bar{t}) - \alpha_i | = \lim_{m\to\infty} \sum_{i=1}^{n} | a_i(t_m) - b_{m,i}(t_m)| \geqslant \epsilon. \tag{3.3}$$

But (3.2) implies that $\alpha_i = a_i(\bar{t})$, $i = 1, 2,..., n$, contradicting (3.3). ∎

COROLLARY. *Let $\mathbf{v}_1, \mathbf{v}_2,..., \mathbf{v}_n$ be elements in $\mathbf{Z}_\infty$ that form a $C(T)$-module Chebyshev system for each $\mathbf{z} \in \mathbf{Z}_\infty$. Given $\mathbf{z} \in \mathbf{Z}_\infty$, let $\mathscr{R}(\mathbf{z})$ denote the singleton contained in $\mathscr{L}_{\mathbf{V}}^1(\mathbf{z})$, where* $\mathbf{V} = \text{span}_{C(T)}\langle \mathbf{v}_1, \mathbf{v}_2,..., \mathbf{v}_n\rangle$. *Then the map $\mathscr{R}: \mathbf{Z}_\infty \to \langle \mathbf{V}, |||\cdot|||_\infty\rangle$ is continuous.*

Proof. Let $\mathbf{z} \in \mathbf{Z}_\infty$ and let $\mathscr{R}(\mathbf{z}) = \sum_{i=1}^{n} a_i\mathbf{v}_i$. Let $\epsilon > 0$. Theorem 5 guarantees that there exists a $\delta > 0$ such that if

$$\left\| \mathbf{z}(t) - \sum_{i=1}^{n} b_i(t)\, \mathbf{v}_i(t) \right\|_t < \rho(t) + \delta \tag{3.4}$$

for all $t \in T$, $b_i \in C(T)$, $i = 1, 2,..., n$, then

$$\sum_{i=1}^{n} | a_i(t) - b_i(t)| < \epsilon \Big/ \sum_{i=1}^{n} ||| \mathbf{v}_i |||_\infty \tag{3.5}$$

for all $t \in T$. Now let $\mathbf{q}$ be any element in $\mathbf{Z}_\infty$ satisfying $|||\, \mathbf{z} - \mathbf{q}\, |||_\infty < \delta/2$. Let $b_i \in C(T)$, $i = 1, 2,..., n$ be such that $\mathscr{R}(\mathbf{q}) = \sum_{i=1}^n b_i \mathbf{v}_i$. Note that

$$\begin{aligned}\left\| \mathbf{z}(t) - \sum_{i=1}^n b_i(t)\, \mathbf{v}_i(t) \right\|_t &\leqslant \| \mathbf{z}(t) - \mathbf{q}(t) \|_t + \left\| \mathbf{q}(t) - \sum_{i=1}^n b_i(t)\, \mathbf{v}_i(t) \right\|_t \\ &\leqslant \| \mathbf{z}(t) - \mathbf{q}(t) \|_t + \left\| \mathbf{q}(t) - \sum_{i=1}^n a_i(t)\, \mathbf{v}_i(t) \right\|_t \\ &\leqslant 2\, \| \mathbf{z}(t) - \mathbf{q}(t) \|_t + \rho(t) \\ &< \delta + \rho(t)\end{aligned}$$

for all $t \in T$. Thus (3.4) implies (3.5) holds for all $t \in T$. Now

$$\begin{aligned}|||\, \mathscr{R}(\mathbf{z}) - \mathscr{R}(\mathbf{q})\, |||_\infty &= \sup_{t \in T} \left\| \sum_{i=1}^n (a_i(t) - b_i(t))\, \mathbf{v}_i(t) \right\|_t \\ &\leqslant \sup_{t \in T} \sum_{i=1}^n |\, a_i(t) - b_i(t) |\, |||\, \mathbf{v}_i\, |||_\infty .\end{aligned}$$

This inequality and (3.5) imply $|||\, \mathscr{R}(\mathbf{z}) - \mathscr{R}(\mathbf{q})\, |||_\infty < \epsilon$. ∎

Throughout the remainder of this section T will denote any compact subset of the real line. Let $\text{span}\langle f_1, f_2, ..., f_m \rangle \subseteq C(T)$ be an m-dimensional Chebyshev subspace. Assume that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are elements in $\mathbf{Z}_\infty$ that form a $C(T)$-module Chebyshev system for each $\mathbf{z} \in \mathbf{Z}_\infty$. As in the Corollary,

$$\mathscr{R}(\mathbf{z}) = \sum_{i=1}^n a_i \mathbf{v}_i , \tag{3.6}$$

$a_i \in C(T)$, $i = 1, 2,..., n$. Then $\mathbf{h}$, where $\mathbf{h}(i) = a_i$, $i = 1, 2,..., n$, is an element of the Banach space $\mathbf{A}_\infty$ defined in Example 1. If $\sum_{j=1}^m c_j(i)\, \mathbf{f}_j(i)$ is constructed as in Example 1, then define $\mathscr{F}(\mathbf{z})$ in $\mathbf{V} = \text{span}_{C(T)}\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle$ by the formula

$$\mathscr{F}(\mathbf{z}) = \sum_{i=1}^n \sum_{j=1}^m c_j(i) f_j \mathbf{v}_i . \tag{3.7}$$

Then (3.7) is the *best product approximation* to $\mathbf{z}$ from $\mathbf{V}$.

We note that the Corollary to Theorem 5 implies that the mapping $\mathscr{F}: \mathbf{Z}_\infty \to (\mathbf{V}, |||\cdot|||_\infty)$ is continuous.

The product approximation (3.7) is the L^∞ product approximation considered by Weinstein [13, p. 183] if $\mathbf{Z}_\infty$ is constructed as in Example 4 (denoted $\mathbf{A}_\infty$ in that example) with $\mathbf{Y}$ being the complex numbers and if $\{g_1, \ldots, g_n\}$ and f of Example 4 are assumed to be real valued.

The L^p product approximations, $1 \leqslant p < \infty$, of [13] are encompassed in (3.7) if $\mathbf{Z}_\infty$ is constructed as in Example 5 and if in Example 1 the L^∞ norm is replaced by the corresponding L^p norm.

We also note that the product approximation continuity theorem in [8, p. 28] is a special case of the continuity of $\mathscr{F}$.

The admissible domains in [13] are based on the somewhat technical Properties 2.4 and 2.5 and on Lemma 2.10 of [13]. In the more general setting of this paper any domain emitting the construction of a continuous field of Banach spaces is admissible. Admissible domains in Theorem 3 and Example 4 are determined by requiring that an appropriate projection mapping be open. For Example 4, Property 2.4 and Lemma 2.10 of [13] imply the openness of the projection mapping. For Example 5 of this paper, Property 2.5 and Lemma 2.10 of [13] imply that (2.19) is a continuous field of Banach spaces. Thus admissible domains in the sense of [13] are admissible domains for the more general Examples 4 and 5 of this paper.

4. CONSLUSIONS

In this paper the approximation of elements of a Banach space $\mathbf{Z}_\infty$ defined by a continuous field of Banach spaces is considered. The approximating space is the $C(T)$ span of a $C(T)$-module Chebyshev system. Product approximation as defined in [12, 13] and subsequently considered in [7, 8] is shown to be a special case of the approximation concepts of this paper. Product approximation in a Banach space defined by a continuous field of Banach spaces incorporates (without additional requirements) product approximation on more complicated domains as examined in [13].

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